The integration inverse method is used in this example. The convergence criterion used in the Newton-Ralphson method is the error $\leq 10^{-6}$. Euler integration is used in the forward simulation with a step size $T/1000$ s. Figure 1 shows the inverse simulation results when $T = 0.1$ s, which is very close to the exact solution sin $2t$. Figure 2 shows the results when $T = 0.01$ s. The solution exhibits the high-frequency oscillations mentioned in the paper (Ref. 1, p. 922). When T is less than 0.005 s, the result becomes erratic and unstable, as shown in Fig. 3.

Conclusion

From the preceding analysis and example, we can conclude that if there are unspecified intermediate variables in the system, the discretization interval T should not be too small. Hence the choice of T is based on two factors:

1) It has to be small enough so that the assumption $u(t) = \text{const}$ in $t_0 \leq t \leq t_0 + T$ is valid.

2) It has to be large enough so that the errors carried by the unspecified intermediate variables are insignificant.

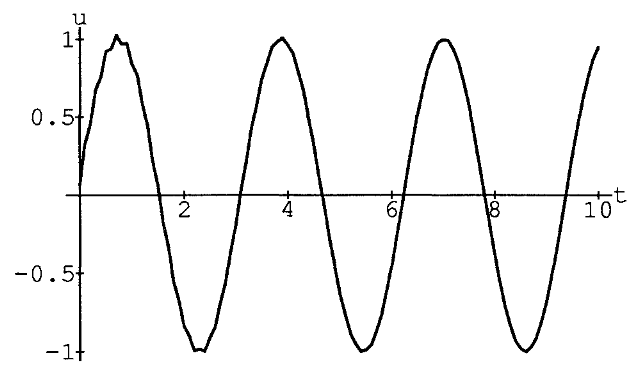

Fig. 1 Result of inverse simulation, $T = 0.1$ s.

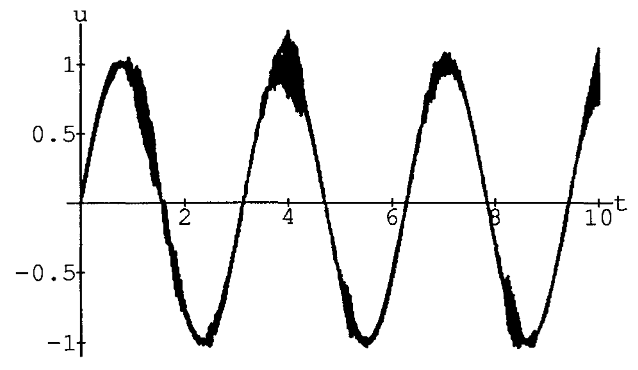

Fig. 2 Result of inverse simulation, $T = 0.01$ s.

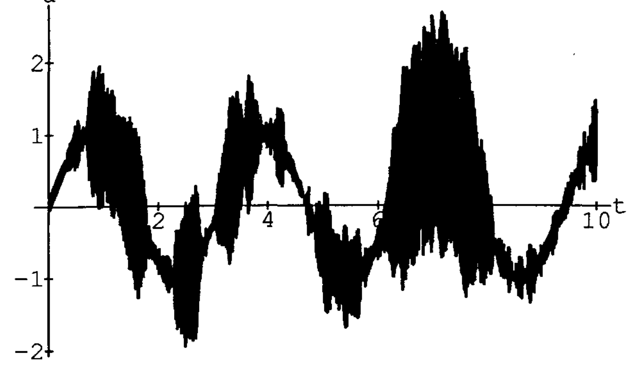

Fig. 3 Result of inverse simulation, $T = 0.005$ s.

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[1]Hess, R. A., Gao, C., and Wang, S. H., "Generalized Technique for Inverse Simulation Applied to Aircraft Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 920–926.

Reply by Authors to Kuo-Chi Lin

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We wish to thank Professor Lin for his interest in our paper and for his instructive example and comments. In Ref. 1, we have attempted to provide a technique for obtaining accurate representations of the control inputs that will force a nonlinear dynamic system to follow a desired trajectory with a fidelity suitable for engineering analyses. The resulting algorithm allows the number of inputs to be equal to or larger than the number of constrained (desired) outputs.

Through a nice example, Professor Lin has demonstrated a phenomenon that we discussed briefly in Ref. 1. Quoting from Ref. 1: "In some of the inverse solutions to be discussed, the control inputs obtained from the simulation exhibited low-amplitude, high-frequency oscillations superimposed on the low-frequency waveform.... These oscillations were filtered by the vehicle dynamics and had minimal impact on the solution quality. They were removed from the (input) solutions to be discussed by use of a fifth-order, low-pass digital filter in the simulation process. The filter had a cutoff frequency of 10 rad/s. The control inputs were passed through the filter twice (forward and backward) to avoid time shifting in the output data."

None of the input solutions in Ref. 1 exhibited the large-amplitude oscillations shown in Fig. 3 of Professor Lin's comment. However, we did obtain oscillations with relatively large amplitudes in a more recent application of the inverse simulation technique.[2] In spite of these oscillations, the filtering technique just described, while admittedly inelegant, allowed us to meet the objective stated in the first paragraph.

In the Discussion section of Ref. 2 we state, "It would be desirable to minimize the oscillatory control inputs which sometimes occur in the inverse simulation. Algorithm modifications which may alleviate this problem are 1) allowing the discretization interval T to be adaptive with respect to the value of the error function $F_E, \ldots$: and 2) improving the accuracy of the Jacobian matrix by perturbing the system with multiple inputs Δu, both positive and negative in sign, and using the average values of the various resulting $\partial y_i/\partial u_j$ as elements of the Jacobian J."

Rather than attempting to adjust T in each of our inverse solutions, we have demonstrated that filtering the control inputs has allowed us to approximate the "true" control inputs with sufficient accuracy for engineering analyses.[1-3] We hasten to point out that, in Refs. 1–3, we always demonstrate the quality of the resulting control input solution by using the filtered control signals as inputs to a forward simulation of the dynamic system in question.

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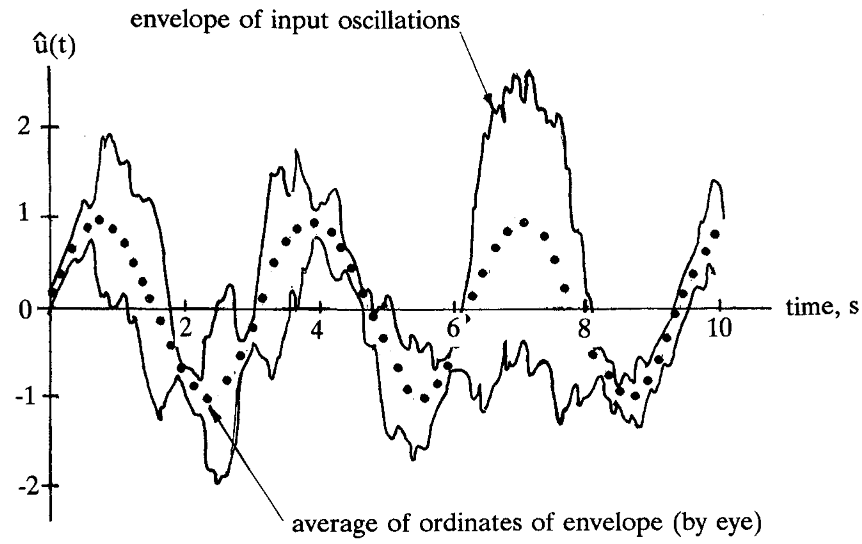

Fig. 1 Re-creation of Professor Lin's Fig. 3.

While agreeing with Professor Lin's caveat regarding the influence of T upon calculated inputs, we take issue with his description of the calculated inputs as "unstable," and by implication, unusable. It has been our experience that the oscillations which appear in the calculated inputs are not unstable, i.e., they do not grow without bound. Indeed, Professor Lin's Figs. 2 and 3 bear this out.

As Professor Lin states, there will always be errors in any numerical technique such as Newton's method. Thus, in our inverse simulation technique, at each $t = kT$, the calculated input $u(kT)$ will be slightly in error. In addition, even if $u(kT)$ were exact, the fact that we are holding $u(kT)$ constant for T, while integrating over the next discretization interval (with an integration step size $\Delta t \ll T$), will always induce errors in the system output $y[(k+1)T]$. However, the inverse solution technique is not blind to these errors. At $t = (k+1)T$, the errors in $y[(k+1)T]$ are incorporated into the calculation for $u[(k+1)T]$, etc. Indeed, it is these corrections that can give rise to the oscillatory behavior just described.

When they occur, these input corrections are truly oscillatory in nature. With $\Delta u_j(kT) \triangleq u_j(kT) - u_j[(k-1)T]$, one finds that at each discretization step T, $\Delta u_j(kT)$ undergoes a sign change as compared to $\Delta u_j[(k-1)T]$. Thus, the $u(kT)$ resulting from the inverse technique can be approximated as

$$u(kT) \approx u_d(kT) + n_u(kT) \tag{1}$$

where $u_d(kT)$ represents the "true" input and $n_u(kT)$ represents a high-frequency oscillatory additive noise. Since Δu_j changes sign at each T, the majority of "power" in each element of the vector $n_u(kT)$ is concentrated around the frequency

$$\omega_0 \approx \pi/T \tag{2}$$

As Professor Lin indicates, the oscillations may be exacerbated by small T. However, as Eq. (2) indicates, as T decreases, the n_u component increases in frequency, and the effect upon the system response is mitigated considerably, i.e., the power in the components of n_u is at a frequency increasingly beyond the bandwidth of the dynamic system in question. This assumes that one has followed the typical rule of thumb in selecting the discretization interval T as[4]

$$2\pi/T \geq 20\omega_B \tag{3}$$

where ω_B is the bandwidth of the system being controlled. This is the case in all of Professor Lin's examples. According to Eq. (2), most of the power in the $n_u(kT)$ portion of $u(kT)$ in Professor Lin's Fig. 3 would be concentrated around the frequency $\omega_0 \approx \pi/T = \pi/0.005 = 628$ rad/s! In his Fig. 2, the frequency would be approximately 314 rad/s. These very high frequencies, of course, explain the blackened areas in the two figures. Now in Professor Lin's Fig. 1, the oscillation frequency would be approximately 31.4 rad/s, and should be visible to the naked eye in the figure. Using an enlarged copy of Professor Lin's Fig. 1, we can just detect a few small-amplitude oscillations near the first peak. A crude measurement of the oscillations visible in the figure indicates an approximate frequency of 35 rad/s.

Referring back to Fig. 3, and recalling that the bandwidth of Professor Lin's second-order dynamic system is approximately 1 rad/s, one can see that the system simply cannot respond to the 628 rad/s oscillations in any significant manner. Thus, if one filters out the $u_n(kT)$ component as we have done, there is little if any change in the system response.

The validity of Eq. (1) and our filtering technique can be demonstrated by another exercise. Figure 1 is a re-creation of Professor Lin's Fig. 3, in which we have traced the envelope of the input oscillations. Using a ruler, we have averaged the ordinates of the upper and lower bounds at a number of time instants across the input duration. The dotted curve results. We have found that this spatial averaging produces time histories that closely resemble those obtained with the more formal temporal filtering process described in the second paragraph. As Fig. 1 indicates, even within the limitations of a 2H pencil and a ruler, one can approximate the "true" input $u_d(t)$ which would force Professor Lin's continuous second-order system to follow the desired output with a fidelity suitable for engineering analyses.

As a final minor point, we were unable to reproduce the relations immediately below Eq. (6) in Professor Lin's comment. Using his Eq. (5), we obtain instead

$$\frac{\partial \hat{u}}{\partial x_1} \approx \frac{\omega_n^2}{2\sigma T} = \frac{\omega_n}{2\zeta T}$$

$$\frac{\partial \hat{u}}{\partial x_0} \approx \frac{-\omega_n^2}{2\sigma T} = \frac{-\omega_n}{2\zeta T}$$

$$\frac{\partial \hat{u}}{\partial \dot{x}_0} \approx \frac{-\omega_n^2}{2\sigma} = \frac{-\omega_n}{2\zeta}$$

In summary, we are in agreement with Professor Lin's basic caveat regarding the size of T, although we disagree about the severity of the problem, given our research objective and our documented results.[1-3] Also, in highly nonlinear, multi-input, multi-output systems, the input oscillation problem may be attributable to more than just the size of T. We have alluded to this in our suggestion regarding improving the accuracy of the Jacobian matrix.[2] Obviously, it would be desirable to eliminate the filtering process in the inverse simulation technique and we feel that more attention needs to be paid to the input oscillation problem. As suggested in Ref. 2, this issue should be addressed in future research. Professor Lin's comment has certainly provided food for thought and we again express our appreciation for his interest and his instructive example.

References

[1]Hess, R. A., Gao, C., and Wang, S. H., "Generalized Technique for Inverse Simulation Applied to Aircraft Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 920–926.

[2]Gao, C., and Hess, R. A., "Inverse Simulation of Large-Amplitude Aircraft Maneuvers," *Journal of Guidance, Control, and Dynamics,* Vol. 16, No. 4, 1993, pp. 733–737.

[3]Hess, R. A., and Gao, C., "A Generalized Algorithm for Inverse Simulation Applied to Helicopter Maneuvering Flight," *Journal of the American Helicopter Society* (to appear).

[4]Franklin, G. F., Powell, J. D., and Workman, M. L., *Digital Control of Dynamic Systems*, 2nd ed., Addison Wesley, Reading, MA, 1990, Chap. 5.

Errata

Averaging of Second-Order Hamiltonian Oscillators with a Slowly Varying Parameter

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[J. Guidance 16(4), pp. 782–784 (1993)]

AN error appears in Eq. (21) and the line immediately following in the Note. The AIAA Editorial Staff regrets this error and any inconvenience it has caused our readers. The correct information appears below:

$$x(u;k) = a - (a - b)\mathrm{sn}^2(u;k) \tag{21}$$

The function $\mathrm{sn}(u;k)$ is the *sine amplitude* with argument $u = \lambda t + u_0$, frequency $\lambda = \sqrt{(a-c)/6}$, and modulus $k^2 = (a - b)/(a - c)$.